UNIMODALITY AND DOMINANCE
FOR SYMMETRIC RANDOM VECTORS

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ABSTRACT. In this paper a notion of unimodality for symmetric random vectors in R^N is developed which is closed under convolution as well as weak convergence. A related notion of stochastic dominance for symmetric random vectors is also studied.

1. **Introduction.** In this paper we explore new notions of unimodality and dominance for random vectors in R^N (for exact definitions see §3). These notions are essentially generalizations of definitions in Sherman [13]. It must be noted, however, that we do not restrict ourselves to random vectors with density as Sherman did; rather, we try to explore our subject in its natural context. In order to accomplish our purpose we need to delve into the interesting topic of logarithmically concave measures, which we do in §2.

In §3 we prove our main results. Namely we show that the class of symmetric unimodal probability measures on R^N is closed under convolution and weak convergence (where unimodality is defined as in Definition 3.1). We also prove some general results about the notion of symmetric dominance for symmetric unimodal random vectors.

In §4 we apply our general results to the study of infinitely divisible random vectors. We give conditions for infinitely divisible random vectors to be symmetric unimodal and as a by-product we see that all symmetric stable random vectors are unimodal. We then make use of semigroup theory in Hille and Phillips [6] to show that if $X(t)$ and $Y(t)$ are homogeneous processes with independent increments with symmetric unimodal Levy-Khinchine measures M_X and M_Y, then $X(t)$ dominates $Y(t)$ for all $t > 0$ if and only if M_X dominates M_Y.

In §5 we compare our notion of unimodality in R^N to three other ways of defining unimodality due to Anderson [1], Olshen and Savage [8], and Ghosh [5].

We end this introduction with a list of basic notation, some definitions and a summary of some required facts about measures and metric spaces.

Notation. For a vector $a \in R^N$ we let $\|a\|$ stand for an arbitrary norm on R^N, while $\|a\|_2$ stands for the usual Euclidean norm $(a \cdot a)^{1/2}$. We let $B(r) = \{y | y \in R^N, \|y\|_2 \leqslant r\}$.

For $U \subset R^N$ we let I_U be the indicator function of U.

For $U \subset R^N, B(U, r) = \{y | y \in R^N, \inf_{x \in U} \|y - x\|_2 \leqslant r\}$.

We let $\mathrm{cl}(U)$, U°, U^c, and ∂U denote the closure of U, the interior of U, the complement of U, and the boundary of U respectively. For $B(1) \subset R^N$ we denote $\partial B(1)$ by S_{N-1}.

We define $\mathrm{diam}(U) = \sup\{\|x - y\|_2; x, y \in U\}$.

We let $\mathrm{span}(U)$ be the affine subspace spanned by U and $\mathrm{conv}(U)$ be the least convex set containing U.

We let $\dim(U) = $ dimension of $\mathrm{span}(U)$.

Finally if U is a convex set and $L = \mathrm{span}(U)$, we let U_L° stand for the interior of U when viewed as a subset of L.

Definitions.

DEFINITION 1.1. If L is a linear subspace of R^N of dimension k, we let Vol_L stand for k-dimensional Lebesgue measure on L, with the normalizing condition that $\mathrm{Vol}_L(B(1)) = \pi^{k/2}/\Gamma(1 + \frac{1}{2}k)$.

If L' is an affine subspace of R^N of the form $L' = L + x$ where L is a linear subspace of R^N we let $\mathrm{Vol}_{L'}(A') = \mathrm{Vol}_L(A)$ where $A' = A + x$ and A is any Borel subset of L.

DEFINITION 1.2. If K is a compact, convex subset of R^N we define the probability measure λ_K on R^N by

(1.2) $$\lambda_K(A) = \mathrm{Vol}_L(A \cap K)/\mathrm{Vol}_L(K)$$

where A is any Borel subset of R^N and L is the affine subspace spanned by K.

Measures and metric spaces.

Let $\mathfrak{M}$ be a metric space with metric d. A measure on $\mathfrak{M}$ is a countably additive $[0, \infty]$ valued set function on the σ-field of Borel subsets of $\mathfrak{M}$, i.e. those sets generated by the open subsets of $\mathfrak{M}$. A probability measure on $\mathfrak{M}$ is a measure on $\mathfrak{M}$ with $\mu(\mathfrak{M}) = 1$.

DEFINITION 1.3. Let $\mathfrak{M}$ be a metric space. Let μ_n be a sequence of bounded measures on $\mathfrak{M}$ and let μ be a bounded measure on $\mathfrak{M}$. We write $\mu_n \to^w \mu$ (and say μ_n converges weakly to μ) if

(1.3) $$\int f \, d\mu_n \to f \, d\mu$$

for all bounded, continuous, real valued functions on $\mathfrak{M}$. If X_n and X are random vectors in R^N we say X_n converges to X in distribution (and write $X_n \to^D X$) if $\mu_n \to^w \mu$ where μ_n is the distribution of X_n (i.e. $\mu_n(A) = P[X_n \in A]$ for any Borel subset of R^N) and where μ is the distribution of X.

It is well known (see Parthasarathy [9]) that if $\mathfrak{M}$ is separable then the set of probability measures on $\mathfrak{M}$ may be made into a separable metric space $(\mathfrak{M}^+, d^+)$ where $d^+(\mu_n, \mu) \to 0$ if and only if $\mu_n \to^w \mu$. If $\mathfrak{M}$ is complete then $\mathfrak{M}^+$ will also be complete. We now mention a useful lemma due to Prokhorov, whose proof can be found in [9, p. 47].

LEMMA 1.1. *Let $(\mathfrak{M}, d)$ be a complete separable metric space. Let $F = \{ \mu_\alpha | \alpha \in A \}$ be a set of probability measures on $\mathfrak{M}$. Then F has compact closure in $(\mathfrak{M}^+, d^+)$ if and only if for all $\varepsilon > 0$ there exists a compact subset K of $\mathfrak{M}$ such that*

$$(1.4) \qquad\qquad \mu_\alpha(K) \geq 1 - \varepsilon$$

for all $\alpha \in A$.

Again let $(\mathfrak{M}, d)$ be a complete separable metric space. Let $(\mathfrak{M}^{++}, d^{++})$ denote the set of probability measures on $\mathfrak{M}^+$ equipped with metric d^{++} which metrizes the notion of weak convergence of probability measures on $\mathfrak{M}^+$. Let F be a set of probability measures on $\mathfrak{M}^+$. A double application of Lemma 1.1 enables us to conclude that F has compact closure in $(\mathfrak{M}^{++}, d^{++})$ if and only if for all $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exists a compact subset K of $\mathfrak{M}$ such that for all $\nu \in F$

$$(1.5) \qquad \nu \{ \mu | \mu \in \mathfrak{M}^+, \mu(K) \geq 1 - \varepsilon_1 \} \geq 1 - \varepsilon_2.$$

If $\mathfrak{M} = R^N$ we shall use the notation $\mathfrak{M}^+ = \mathfrak{M}_N^+$ and $\mathfrak{M}^{++} = \mathfrak{M}_N^{++}$. If $\mathfrak{M} = L$, a linear subspace of R^N, we shall write $\mathfrak{M}^+ = \mathfrak{M}_L^+$ and $\mathfrak{M}^{++} = \mathfrak{M}_L^{++}$.

DEFINITION 1.4. Let $\mu_1, \mu_2 \in \mathfrak{M}^+$. We define the total variation norm between μ_1 and μ_2 (written $\| \mu_1 - \mu_2 \|_{\text{t.v.}}$) to be $\sup_A |\mu_1(A) - \mu_2(A)|$ where the sup is taken over the family of all Borel subsets A of $\mathfrak{M}$. We write $\mu_n \to^{\text{t.v.}} \mu$ if $\| \mu_n - \mu \|_{\text{t.v.}} \to 0$. It is well known that $\mu_n \to^{\text{t.v.}} \mu$ implies $\mu_n \to^w \mu$. It is also well known that $(\mathfrak{M}^+, \| \ \|_{\text{t.v.}})$ is a complete metric space, even if $\mathfrak{M}$ is not complete.

Suppose now that L is a linear subspace of R^N. If h is a Borel measurable real valued function on L we define $\|h\|_L = \int_L |h(x)| \, d \operatorname{Vol}_L(x)$. We shall let $\mathcal{L}^1(\operatorname{Vol}_L)$ denote the set of all Borel measurable functions h on L with $\|h\|_L < \infty$. If μ is a probability measure on L which is absolutely continuous with respect to Vol_L and has Radon-Nikodym derivative f with respect to Vol_L, we shall write $d\mu = f \, d \operatorname{Vol}_L$ or $d\mu / d \operatorname{Vol}_L = f$. It is well known that if μ, ν are probability measures on L with $d\mu = f \, d \operatorname{Vol}_L$ and $d\nu = g \, d \operatorname{Vol}_L$, then $\| \mu - \nu \|_{\text{t.v.}} = \frac{1}{2} \|f - g\|_L$.

DEFINITION 1.5. Let $(\mathfrak{M}, d)$ be a complete separable metric space and let μ be a measure on $\mathfrak{M}$. We define the support of μ (written $\operatorname{supp}(\mu)$) to be the

smallest closed subset D of $\mathfrak{M}$ such that $\mu(D^c) = 0$. Equivalently, supp(μ) is the set of all points x in $\mathfrak{M}$ such that $\mu(x + B(a)) > 0$ for all $a > 0$. (See [9, p. 29].)

We end the introduction with a well-known lemma about weak convergence of measures (see [9, p. 40] for the proof).

LEMMA 1.2. *Let* ($\mathfrak{M}$, *d*) *be a metric space. Let* μ_n *be a sequence of bounded measures on* $\mathfrak{M}$ *with* $\mu_n \rightarrow^w \mu$. *Then*:

(1) *For any closed set C in* $\mathfrak{M}$ *we have* $\lim \sup_{n\to\infty} \mu_n(C) \leqslant \mu(C)$.

(2) *For any Borel subset A of* $\mathfrak{M}$ *with* $\mu(\partial A) = 0$ *we have* $\lim_{n\to\infty} \mu_n(A) = \mu(A)$.

2. Log-concave measures. In this section we present the main facts about an important class of probability measures on R^N.

DEFINITION 2.1. We say that a probability measure μ on R^N is logarithmically concave if for all open, convex sets $C, D \subset R^N$ and $t \in (0, 1)$ we have

(2.1) $$\mu(tC + (1 - t)D) \geqslant \left(\mu(C)^t\right)\left(\mu(D)^{1-t}\right).$$

It is easy to deduce that (2.1) is also valid for all closed, convex sets C, D, since any closed convex set is the limit of a decreasing sequence of open convex sets. An easy consequence of (2.1) is the inequality

(2.2) $$\mu\big(C - (tx + (1 - t)y)\big) \geqslant \left(\mu(C - x)^t\right)\left(\mu(C - y)^{1-t}\right),$$

valid for all $x, y \in R^N$. A direct consequence of (2.2) is the fact that supp(μ) is a closed convex set. If we let $L = \text{span}(\text{supp}(\mu))$ it is natural to conjecture that $d\mu = f\, d\,\text{Vol}_L$ for some logarithmically concave function f on L. (A function f is said to be log-concave if $\log f$ has values in $[-\infty, +\infty)$ and is concave.) This conjecture has recently been proved in [2, p. 123]. The converse to this conjecture was demonstrated by Prekopa [10]. We combine these results to get the following key lemma.

LEMMA 2.1. *Let μ be a logarithmically concave measure on R^N. Then there exists a linear subspace L of R^N and a log-concave density function f such that $d\mu = f\, d\,\text{Vol}_L$. Conversely if L and a log-concave density function f are given, then the measure μ defined by $d\mu = f\, d\,\text{Vol}_L$ is logarithmically concave.*

COROLLARY 2.1. *For any compact, convex, symmetric set K and any symmetric open, convex set C we have $\lambda_K(C - x) \leqslant \lambda_K(C)$.*

PROOF. The measure λ_K has log-concave density $\text{Vol}_L(K)^{-1}I_K$ where $L = \text{span}(K)$, hence by Lemma 2.1, λ_K is a log-concave measure. Now use (2.2). Q.E.D.

LEMMA 2.2. *Suppose* ($\mu_n | n \geqslant 1$) *is a sequence of log-concave probability measures with* $\mu_n \rightarrow^w \mu$. *Then μ is also a log-concave probability measure.*

PROOF. We wish to show that μ satisfies (2.1). In fact whenever C, D are closed convex sets with nonempty interior and $\mu(\partial C) = \mu(\partial D) = 0$, then (2.1) is satisfied since in that case

$$\mu(C) = \lim_{n \to \infty} \mu_n(C) \quad \text{and} \quad \mu(D) \lim_{n \to \infty} \mu_n(D)$$

by Lemma 1.2. To demonstrate (2.1) for general open convex sets C and D, we note that any open convex set A is the limit of an increasing sequence A_n, of closed convex sets with nonempty interior which are chosen so that $\mu(\partial A_n) = 0$. Q.E.D.

LEMMA 2.3. *Let μ_1 and μ_2 be log-concave probability measures on R^N. Then $\mu_1 * \mu_2$ is also a log-concave probability measure on R^N.*

PROOF. Let $C_i = \text{supp}(\mu_i)$ for $i = 1, 2$. Translating μ_i by vectors x_i if necessary, we may suppose $0 \in C_1 \cap C_2$. Let $L_i = \text{span}(C_i)$ and let $L_i^\perp$ be the linear subspaces orthogonal to L_i so that $R^N = L_i \oplus L_i^\perp$ for $i = 1, 2$. Let

$$g_{i,n}(y_i) = \frac{1}{\text{Vol}_{L_i^\perp}(B(1/n))} I_{B(1/n) \cap L_i^\perp}(y_i) \quad \text{for} \quad y_i \in L_i^\perp$$

and let

$$f_{i,n}(x_i) = \frac{1}{\text{Vol}_{L_i}(B(1/n))} \mu_i(B(1/n) - x_i) \quad \text{for} \quad x_i \in L_i.$$

Let $\nu_{i,n}$ be the probability measure on R^N with density $f_{i,n} g_{i,n}$ with respect to $\text{Vol}_L = \text{Vol}_{L_i} \times \text{Vol}_{L_i^\perp}$. It is proved in [10] as well as [3] that $\nu_{1,n} * \nu_{2,n}$ is a log-concave probability measure.

It is clear that $\nu_{1,n} \to^w \mu_1$ and $\nu_{2,n} \to^w \mu_2$ as $n \to \infty$. We conclude that $\nu_{1,n} * \nu_{2,n} \to^w \mu_1 * \mu_2$ and $\mu_1 * \mu_2$ is a log-concave measure, since log-concavity is preserved under weak limits. Q.E.D.

LEMMA 2.4. *Let μ be a log-concave measure on R^N. Let L be a linear subspace and let Π_L stand for the orthogonal projection operator onto L. Then the measure $\mu \circ \Pi_L$ is a log-concave measure on L.*

PROOF. If C, D are open convex subsets of L then $\Pi_L^{-1}(C)$ and $\Pi_L^{-1}(D)$ are open convex subsets of R^N. Since μ is log-concave, (2.1) is satisfied for $\Pi^{-1}(C)$, $\Pi^{-2}(D)$; it follows that (2.1) is satisfied for $\mu \circ \Pi_L^{-1}$ and C, D. Q.E.D.

DEFINITION 2.2. Let L be a linear subspace of R^N. We shall denote by T_L the set of all log-concave, symmetric nonnegative functions f on L with $\int_L f \, d \, \text{Vol}_L = 1$, and we shall denote by T_L^* the set of all probability measures μ on L of the form $d\mu = f \, d \, \text{Vol}_L$ where $f \in T_L$.

COROLLARY 2.3. *The set T_L is complete separable metric space when equipped with the metric $\| \cdot \|_L$ and when functions which agree a.e. Vol_L are identified.*

PROOF. Suppose $\|f_n - f_m\|_L \to 0$ as $n, m \to \infty$, where the sequence $f_n \in T_L$. Then $\|\mu_n - \mu_m\|_{\text{t.v.}} \to 0$, where μ_m is a sequence of probability measures on L defined by $d\mu_n = f_n \, d\,\text{Vol}_L$. Now this implies the existence of some probability measure μ on L such that $\|\mu_n - \mu\|_{\text{t.v.}} \to 0$ as $n \to \infty$. In particular, $\mu_n \to^w \mu$, hence by Lemma 2.2, μ is a log-concave measure on R^N. Now μ has density with respect to Vol_L since $\|\mu_n - \mu\|_{\text{t.v.}} \to 0$; also μ is clearly symmetric. We can now use Lemma 2.1 to conclude that $d\mu/d\,\text{Vol}_L = f$ where $f \in T_L$.

We have just proved completeness of T_L^*. The proof of separability is standard.   Q.E.D.

3. Symmetric unimodal random vectors. In this section we shall introduce our notion of unimodality for symmetric random vectors in R^N, and we shall prove a few fundamental results about this notion.

We shall often be dealing with sets $\mathbb{S}$ which consist of a subset of the collection of all probability measures on some metric space $\mathfrak{M}$ and we shall need to have a uniform convention for the concept of Borel subsets of $\mathbb{S}$. Our convention shall be to endow $\mathbb{S}$ with the Borel structure consisting of the least σ-field containing all those subsets U of $\mathbb{S}$ which are open in the topology of weak convergence of probability measures.

DEFINITION 3.1. Let L be a linear subspace of R^N. We shall let W_L stand for the set of all probability measures on L of the form λ_K where K is a compact symmetric subset of L. We give W_L the Borel structure mentioned above.

We let U_L stand for all the measures μ on R^N which are of the form

$$(3.1) \qquad\qquad \mu = \int_{W_L} \lambda_K \, d\nu(\lambda_K)$$

where ν is a measure on W_L. We call ν the mixing measure for μ. (We shall later show that W_L is a Borel measurable subset of $\mathfrak{M}_L^+$.) We note that μ in (3.1) is a probability measure if and only if ν is a probability measure.

We shall use the notation $W_{R^N} = W_N$ and $U_{R^N} = U_N$. We shall call a random vector in R^N, symmetric unimodal if its distribution is in U_N. We note that if μ is a symmetric probability measure on R which is unimodal in the standard 1-dimensional sense, then by Feller [4, p. 158] we have $\mu \in U_1$.

LEMMA 3.1. *Let L be a linear subspace of R^N and let μ be a log-concave, symmetric, probability measure on L with $d\mu/d\,\text{Vol}_L = f$ where $f \in T_L$. Then $\mu \in U_L$.*

PROOF. Let $D_s = \{y \,|\, f(y) \geqslant s\}$ for $s > 0$. We have $f(x) = \int_0^\infty I_{D_s}(x) \, ds$; hence it follows that for any Borel subset A of L,

$$(3.2) \qquad \int_A f(x) \, d\operatorname{Vol}_L(x) = \int_0^\infty \operatorname{Vol}_L(A \cap D_s) \, ds.$$

Now D_s is a bounded convex subset of L for $s > 0$ and it is clear that $\operatorname{Vol}_L(D_s) = \operatorname{Vol}_L(K_s)$ for all s, where $K_s = \operatorname{cl}(D_s)$. It follows that

$$\operatorname{Vol}_L(A \cap D_s) = \operatorname{Vol}_L(A \cap K_s)$$

for all s and we can rewrite (3.2) as

$$(3.3) \qquad \int_A f(x) \, d\operatorname{Vol}_L(x) = \int_0^\infty \operatorname{Vol}_L(A \cap K_s) \, ds.$$

Now let Φ be the measurable map from $(0, \infty)$ into W_L defined by $\Phi(s) = \lambda_{K_s}$ and let $\nu' = \mathcal{L} \circ \Phi^{-1}$ where $\mathcal{L}$ is Lebesgue measure on $(0, \infty)$. We rewrite (3.3) as

$$(3.4) \qquad \int_A f(x) \, d\operatorname{Vol}_L(x) = \int_{W_L} \lambda_K(A) \operatorname{Vol}_L(K) \, d\nu'(\lambda_K).$$

Finally we let the measure ν on W_L be defined by the equation

$$(3.5) \qquad \nu(B) = \int_B \operatorname{Vol}_L(K) \, d\nu'(\lambda_K)$$

where B is any Borel subset of W_L. It is clear that we can rewrite (3.4) as

$$(3.6) \qquad \int_A f(x) \, d\operatorname{Vol}_L(x) = \int_{W_L} \lambda_K(A) \, d\nu(\lambda_K);$$

hence we conclude that $\mu \in U_L$ with mixing measure ν. Q.E.D.

Notation. Let μ and ν be as in Lemma 3.1. We shall denote ν by $J(\mu)$. We claim that J is a well-defined map from T_L^* into W_L^+. In fact suppose

$$d\mu = f_1 \, d\operatorname{Vol}_L = f_2 \, d\operatorname{Vol}_L$$

where $f_1 = f_2$ a.e. (Vol_L) and $f_1 = f_2 \in T_L$. It is then obvious that the two compact convex sets $\operatorname{cl}\{y|f_1(y) \geqslant s\}$ and $\operatorname{cl}\{y|f_2(y) \geqslant s\}$ are equal for all $s \geqslant 0$.

We shall study the measurability properties of the map J in the following lemma.

LEMMA 3.2. *The map* $J: T_L^* \to W_L^+$ *is measurable.*

PROOF. We must show that for any Borel set C of W_L^+, $\{\mu | J(\mu) \in C\}$ is a Borel subset of T_L^*. It suffices to consider sets C of the form

$$(3.7) \qquad C = \left\{ \nu \,\middle|\, \int_{W_L} k(\lambda_k) \, d\nu(\lambda_K) \geqslant r \right\}$$

where h is a bounded continuous function on W_L and $r \in R$. We can assume $h(\lambda_K) = \int_L g \, d\lambda_K$ where g is a bounded continuous function on L.

We know that any $\mu \in T_L^*$ is of the form $d\mu = f_\mu \, d \, \mathrm{Vol}_L$ where f_μ is a logarithmically concave density function. Let us set $K_{\mu,s} = \{y \,|\, f_\mu(y) > s\}$. We now compute

$$(3.8) \qquad \int_{W_L} h(\lambda_K) \, d\nu(\lambda_K) = \int_{W_L} \left(\int_L g \, d\lambda_K \right) \mathrm{Vol}_L(K) \, d\nu'(K),$$

whence

$$(3.9) \qquad \int_{W_L} h(\lambda_K) \, d\nu(\lambda_K) = \int_{W_L} \int_K g \, d \, \mathrm{Vol}_L \, d\nu'(K).$$

Remember now that $\nu' = \mathcal{L} \circ \Phi_\mu^{-1}$ where $\Phi_\mu(s) = \lambda_{K_{\mu,s}}$; then the last expression becomes $\int_0^\infty (\int_{K_{\mu,s}} g \, d \, \mathrm{Vol}_L) \, ds$.

It follows that

$$(3.10) \qquad \{\mu \,|\, J(\mu) \in C\} = \left\{ \mu \,\Big|\, \int_0^\infty \left(\int_{K_{\mu,s}} g \, d \, \mathrm{Vol}_L \right) ds > r \right\}.$$

Using (3.2) we get

$$\{\mu \,|\, J(\mu) \in C\} = \left\{ \mu \,\Big|\, \int_L g(x) f_\mu(x) \, d \, \mathrm{Vol}_L(x) > r \right\}.$$

Hence,

$$(3.11) \qquad \{\mu \,|\, J(\mu) \in C\} = \left\{ \mu \,\Big|\, \int_L g \, d\mu > r \right\}.$$

The last set is clearly a Borel subset of T_L^*.   Q.E.D.

COROLLARY 3.1. W_L is a Borel subset of $\mathfrak{M}_L^+$.

PROOF. We have $W_L \subset T_L^* \subset \mathfrak{M}_L^+$.
Using Corollary 2.3 we know that T_L^* is a complete metric space under the total variation metric; also the imbedding $T_L^* \subset \mathfrak{M}_L^+$ is continuous where T_L^* is given the topology determined by the total variation metric while $\mathfrak{M}_L^+$ is given the topology determined by weak convergence of probability measures. We conclude by a theorem of Kuratowski [7, p. 21] that T_L^* is a Borel subset of $\mathfrak{M}_L^+$.

Consider now the set $H \subset W_L^+$ defined by $H = \{\mu \,|\, \mu$ is an atomic probability measure on $W_L\}$. Now W_L is separable under the topology of weak convergence; hence there exists a sequence A_n of Borel subsets of W_L such that $H = \{\mu \,|\, \mu \in \mathfrak{M}_L^+, \mu(A_n) = 0$ or 1 for all $n\}$. We conclude H is a Borel subset of W_L^+. Noting $W_L = J^{-1}(H)$, by Lemma 3.2 we conclude that W_L is a Borel subset of T_L^*. Since T_L^* is a Borel subset of $\mathfrak{M}_L^+$, we are done. Q.E.D.

The next theorem is one of the fundamental results in this paper, namely that symmetric unimodal probability measures are closed under weak convergence. It is easy to prove in 1 dimension; its difficulty in N dimensions is essentially due to the fact that $\overline{W}_N$, the closure of W_N in $\mathfrak{M}_N$ in the topology of weak convergence, contains W_N properly. The following example will clarify things.

EXAMPLE 3.1. In R^2 let

$$K_n = \mathrm{conv}\big(\{(1, 0), (-1, 0), (0, 1/n), (0, -1/n)\}\big).$$

A simple computation shows that $\lambda_{K_n} \to^w \mu$ where μ is the probability measure on $L = \{(x, y)|y = 0\}$ whose density is $|1 - x|$ on $K = \{(x, y)|x \in [-1, 1], y = 0\}$ with respect to Vol_L.

THEOREM 3.1. *The class of symmetric unimodal probability measures on R^N is closed under weak convergence.*

PROOF. Suppose μ_n is a sequence of probability measures in U_N with $\mu_n \to^w \mu$ where μ is a probability measure on R^N. By Lemma 1.1 we know that for any $\theta, \delta > 0$ there exists n so large that $\mu_n(B(r)) > 1 - (\theta/\delta)$ for all n as well as $\mu(B(r)) > 1 - \theta/\delta$. Now we can write

$$(3.12) \qquad \mu_n = \int_{W_N} \lambda_K \, d\nu_n(\lambda_K)$$

for some probability measure ν_n on W_N. We conclude that $\nu_n\{\lambda_K|\lambda_K(B(r))^c > \delta^{-1}\} < \theta$. Let $\overline{W}_N$ stand for the closure of W_N in $\mathfrak{M}_N$. Applying (1.5) we conclude immediately that the set $\{\nu_n|n \geq 1\}$ has compact closure in $(\overline{W}_N)^+$, so by passing to a subsequence if necessary, we know there exists $\overline{\nu} \in (\overline{W}_N)^+$ such that $\nu_k \overset{w}{\to} \overline{\nu}$. This means that

$$(3.13) \qquad \int_{\overline{W}_n} f \, d\nu_n \to \int_{\overline{W}_N} f \, d\overline{\nu}$$

for all bounded continuous real valued functions f on $\overline{W}_N$. Now $\overline{W}_N$ is clearly a measurable subset of T_N^* (since it is a complete separable metric space and the theorem of Kuratowski [9, p. 21] applies), so we conclude by Lemma 3.2 that the restriction $J: \overline{W}_N \to W_N^+$ is a measurable map. It follows that any probability measure Q on $\overline{W}_N$ induces a probability measure $Q \circ J^{-1}$ on W_N^+. In (3.13) let us take $f(\alpha) = \int_{R^N} g(x) \, d\alpha(x)$ where $\alpha \in \overline{W}_N$ and g is a bounded continuous real function on R^N. We can rewrite (3.13) as

$$(3.14) \qquad \lim_{n\to\infty} \int_{W_N^+} \left[\int_{W_N} \left\{ \int_{R^N} g \, d\lambda_K \right\} d\beta(\lambda_K) \right] d\nu_n \circ J^{-1}(\beta)$$

$$= \int_{W_N^+} \left[\int_{W_N} \left\{ \int_{R^N} g \, d\lambda_K \right\} d\beta(\lambda_K) \right] d\overline{\nu} \circ J^{-1}(\beta).$$

It follows that for all such g,

$$(3.15) \quad \int_{R^N} g(x)\, d\mu(x) = \int_{W_N^+} \left[\int_{W_N} \left\{ \int_{R^N} g\, d\lambda_K \right\} d\beta\, (\lambda_K) \right] d(\bar{\nu} \circ J^{-1})(\beta).$$

We now define $\nu \in W_N^+$ by the equation

$$(3.16) \quad \int_{W_N} h(\lambda_K)\, d\nu(\lambda_K) = \int_{W_N^+} \left[\int_{W_N} \left\{ \int_{R^N} g\, d\lambda_K \right\} d\beta\, (\lambda_K) \right] d(\bar{\nu} \circ J^{-1})(\beta),$$

valid for all bounded continuous real valued functions on W_N of the form $h(\lambda_K) = \int_{R^N} g\, d\lambda_K$. Putting (3.13)–(3.16) together we conclude that $\mu = \int_{W_N} \lambda_K\, d\nu(\lambda_K)$.   Q.E.D.

We now prove that symmetric unimodal probability measures on R^N are closed under convolution. This closure property is a strong justification for our belief that we are working with the proper notion of unimodality for symmetric random vectors in R^N. It is well known to be valid in one dimension (note however that the convolution of nonsymmetric unimodal probability measures is not necessarily unimodal as shown in [4, p. 168]).

THEOREM 3.2. *Let* $\mu_1, \mu_2 \in U_N$. *Then* $\mu_1 * \mu_2 \in U_N$.

PROOF. We can write $\mu_i = \int_{W_N} \lambda_K\, d\nu_i(\lambda_K)$ where ν_i are probability measures on W_N for $i = 1, 2$. We conclude that

$$(3.17) \quad \mu_1 * \mu_2 = \int \lambda_{K_1} * \lambda_{K_2}\, d\nu_1(\lambda_{K_1}) \times \nu_2(\lambda_{K_2}).$$

Now $\lambda_{K_1} * \lambda_{K_2}$ is a symmetric log-concave probability measure on R^N by Lemma 2.6, hence $\lambda_{K_1} * \lambda_{K_2} \in U_N$, so we can write

$$(3.18) \quad \lambda_{K_1} * \lambda_{K_2} = \int_{W_N} \lambda_K\, d\nu_{K_1, K_2}(\lambda_K)$$

for some probability measure ν_{K_1, K_2} on W_N. Finally we can write $\mu_1 * \mu_2 = \int_{W_N} \lambda_K\, d\nu(\lambda_K)$ where ν is the probability measure on W_N defined by

$$(3.19) \quad \nu(B) = \int_{W_n \times W_n} \nu_{K_1, K_2}(B)\, d\nu_1(\lambda_{K_1}) \times d\nu_2(\lambda_{K_2}),$$

for all Borel subsets B of W_N.   Q.E.D.

The following closure property is an easy corollary to Lemma 2.7. We state it for completeness, but omit the proof.

THEOREM. 3.3. *Let* $\mu \in U_N$. *Let* L *be a linear subspace of* R^N *and let* Π_L *denote the orthogonal projection operator onto* L. *Then the measure* $\mu \circ \Pi_L^{-1}$ *is in* U_L.

An important corollary of the last theorem is that all marginals of a symmetric unimodal measure on R^N are also symmetric unimodal measures.

We now turn to the notion of dominance.

DEFINITION 3.3. Let μ_1 and μ_2 be measures on R^N. We say that μ_1 symmetrically dominates μ_2 (and write $\mu_1 \succ \mu_2$) if we have

$$(3.20) \qquad \mu_1(K^c) \geqslant \mu_2(K^c)$$

for all compact, convex, symmetric subsets of R^N. If X_1 and X_2 are random vectors in R^N with distribution μ_1 and μ_2, respectively, we write $X_1 \succ X_2$ if $\mu_1 \succ \mu_2$.

Let us denote by $\mathcal{F}_N$ the class of continuous, symmetric log-concave functions f on R^N with values in $[0, \infty)$ such that $f(0) = 1$ and $\lim_{\|x\|_2 \to \infty} f(x) = 0$.

LEMMA 3.3. *Let μ_1 and μ_2 be measures on R^N with $\mu_i(D^c) < \infty$, for any Borel measurable set D containing 0 in its interior. Then the condition that $\mu_1 \succ \mu_2$ is equivalent to the condition that*

$$(3.21) \qquad \int_{R^N} (f(0) - f(x)) \, d\mu_1(x) \geqslant \int_{R^N} (f(0) - f(x)) \, d\mu_2(x)$$

for all $f \in \mathcal{F}_N$.

PROOF. Assume first that $\mu_1 \succ \mu_2$. For $f \in \mathcal{F}_N$ let $f_n = \sum_1^n (k/n) I_{\{x | f(x) > k/n\}}$. Now f_n decreases to f and (3.20) holds for f_n. We conclude that (3.20) holds for f.

Arguing backwards we now assume (3.21) for all $f \in \mathcal{F}_N$. Let K be a compact, symmetric convex subset of R^N containing 0 in its interior. Let $D_s = \partial(sK)$. The sets D_s are all disjoint, hence except for s in a countable set N we have $(\mu_1 + \mu_2)(D_s) = 0$.

Define for $\theta > 0$,

$$f_{\theta,s}(x) = \frac{1}{\mathrm{Vol}_L(B(\theta))} \, \mathrm{Vol}_L\big((B(\theta) - x) \cap sK\big).$$

It is clear that $f_{\theta,s} \in \mathcal{F}_N$ and that $f_{\theta,s}(x) = 1$ if $x \in (sK)^\circ$, for θ sufficiently small. Also for $x \not\in sK$, it is clear that $f_{\theta,s}(x) = 0$ for θ sufficiently small. Now we let $\theta \to 0$ in (3.22) (and use the Lebesgue dominated convergence theorem), in the following inequality,

$$(3.22) \qquad \int_{R^N} (f_{\theta,s}(0) - f_{\theta,s}(x)) \, d\mu_1(x) \geqslant \int_{R^N} (f_{\theta,s}(0) - f_{\theta,s}(x)) \, d\mu_2(x),$$

to get that $\mu_1((sK)^c) \geqslant \mu_2((sK)^c)$ for $s \not\in N$. Letting $s \to 1$, $s \not\in N$, we then get $\mu_1(K^c) \geqslant \mu_2(K^c)$.

If K has an empty interior, we apply the previous argument to get $\mu_1(B(K, \delta)^c) \geqslant \mu_2(B(K, \delta)^c)$ for any $\delta > 0$. Letting $\delta \to 0$ we conclude $\mu_1(K^c) \geqslant \mu_2(K^c)$. Q.E.D.

LEMMA 3.4. *Suppose* $\mu \in U_N$ *and that* μ_1 *and* μ_2 *are two measures on* $R^{N'}$ *with* $\mu_1 \succ \mu_2$. *Then* $\mu \times \mu_1 \succ \mu \times \mu_2$.

PROOF. It is sufficient to prove the lemma for $\mu = \lambda_K \in W_N$, since the general result will follow by mixing. Let h be in $\mathcal{F}_{N+N'}$. We compute

$$(3.23) \quad \int_{R^N \times R^{N'}} h(0_N, 0_{N'}) - h(x, y) \, d\lambda_K(x) \, d\mu_i(y)$$
$$= \int_{R^{N'}} f(0_{N'}) - f(y) \, d\mu_i(y)$$

where 0_N and $0_{N'}$ stand for the zero vectors in R^N and $R^{N'}$, respectively, while

$$(3.24) \qquad f(y) = \int_{R^N} h(x, y) \, d\lambda_K(x).$$

Now f is a log-concave function on $R^{N'}$ by Prekopa [10]; it is obviously continuous symmetric and satisfies $\lim_{\|x\|_2 \to \infty} f(x) = 0$. Using (3.21) we get that

$$\int_{R^N \times R^{N'}} (h(0) - h(z))(d\lambda_K \times d\mu_1)(z)$$
$$\geq \int_{R^N \times R^{N'}} (h(0) - h(z))(d\lambda_K \times d\mu_2)(z). \qquad \text{Q.E.D.}$$

COROLLARY 3.2. *Suppose* $(\mu_i \times \mu_i')$ *for* $i = 1, 2$, *are measures on* $R^N \times R^{N'}$. *Suppose* $\mu_1 \succ \mu_2$, $\mu_1' \succ \mu_2'$ *and that* $\mu_1 \in U_N$, $\mu_2' \in U_{N'}$. *Then* $(\mu_1 \times \mu_1') \succ (\mu_2 \times \mu_2')$.

PROOF. By Lemma 3.4 we know that $(\mu_1 \times \mu_1') \succ (\mu_1 \times \mu_2')$ and that $(\mu_1 \times \mu_2') \succ (\mu_2 \times \mu_2')$. We conclude that $(\mu_1 \times \mu_1') \succ (\mu_2 \times \mu_2')$. Q.E.D.

The following corollary is a generalization of Lemma 3 in Sherman [13].

COROLLARY 3.3. *Suppose* $(\mu_i \times \mu_i')$ *for* $i = 1, 2$, *satisfy the hypothesis of Corollary* 3.2, *where* $N \equiv N'$. *Then* $\mu_1 * \mu_1' \succ \mu_2 * \mu_2'$.

PROOF. Let $Q: R^N \times R^N \to R^N$ stand for the addition operator $Q(x, y) = x + y$. If $f \in \mathcal{F}_N$ we note that $f \circ Q$ is in $\mathcal{F}_{N+N}$. The rest of the proof is straightforward using Corollary 3.2. Q.E.D.

4. Applications to infinitely divisible symmetric random vectors. In this section we study the notions of symmetric unimodality and symmetric dominance for infinitely divisible random vectors.

DEFINITION 4.1. A symmetric random vector X in R^N is infinitely divisible if there exists a time homogeneous stochastic process $(X(t)|t > 0)$ with symmetric independent increments such that $X(0) = 0$ a.s. and $X(1) = X$ a.s.

It is well known that we can write the characteristic function $\phi_X(v) = E(e^{i(v \cdot X)})$ of the random vector X in the form (due to Levy and Khinchine)

$$(4.1) \qquad \log(\phi_X(v)) = \int_{R^N} (\cos(v \cdot y) - 1) \, dM_X(y) - q_X(v)$$

where M_X is a symmetric measure on R^N with $M_X(\{0\}) = 0$, $M_X(B(a)^c) < \infty$ and $\int_{B(a)} (\|y\|_2)^2 \, dM_X(y) < \infty$ for all $a > 0$; while $q_X(y) = (B_X y \cdot B_X y)$ for some symmetric matrix B_X. (Note that the conditions on M_X ensure that $\int \|y - x\|^2 / (1 + \|y - x\|^2) \, dM_X(y) < \infty$ for all $x \in R^N$.) We shall assume $q_X \equiv 0$, unless otherwise stated (no Gaussian part). The reader is referred to [9] for a derivation of (4.1).

Let $C_\infty(R^N)$ denote the set of all bounded infinitely differentiable real valued functions on R^N. Let us define the operator $T^t \colon C_\infty(R^N) \to C_\infty(R^N)$ by

$$(4.2) \qquad T^t f(x) = \int_{R^N} f(x - y) \, d\mu^t(y),$$

where μ^t is the distribution of $X(t)$. The family $(T^t | t > 0)$ is a semigroup of operators and has an infinitesimal generator A defined by

$$(4.3) \qquad Af = \lim_{t \to 0} \frac{1}{t} (T^t - I) f$$

for any f for which this limit exists in the uniform topology on $C_\infty(R^N)$. The class of such functions f is a dense linear subspace W of $C_\infty(R^N)$ (see [6]). For $y \in R^N$, let

$$\tau(y) = \left\{ \begin{array}{ll} y & \text{if } \|y\|_2 < 1, \\ y/\|y\|_2 & \text{if} \|y\|_2 > 1. \end{array} \right\}.$$

It can also be shown that

$$(4.4) \qquad Af(x) = \int_{R^N} (f(x) - f(x - y)) - (f'(x) \cdot \tau(y)) \, dM_X(y)$$

for any $f \in W$. (See [4, p. 301].) If we set $x = 0$ in (4.4) we get that

$$(4.5) \qquad \begin{aligned} &\lim_{t \to 0} \frac{1}{t} \int_{R^N} (f(0) - f(-y)) \, d\mu^t(y) \\ &\qquad = \int_{R^N} (f(0) - f(-y)) - (f'(0) \cdot \tau(y)) \, dM_X(y). \end{aligned}$$

So we conclude

$$(4.6) \qquad \lim_{t \to 0} \frac{1}{t} \int_{R^N} (f(0) - f(-y)) \, d\mu^t(y) = \int_{R^N} (f(0) - f(-y)) \, dM_X(y)$$

for any $f \in W$, where we have used the fact that M_X is a symmetric measure to get the last equality. For the next lemma we need the notation $\nu^s = M_X * \mu^s$.

LEMMA 4.1. *If f is a bounded measurable function which is constant in a neighbourhood of* 0, *then*

$$(4.7) \quad \int_{R^N} (f(0) - f(-x)) \, d\mu^t(x) = \int_0^t \left\{ \int_{R^N} (f(0) - f(-y)) \, d\nu^s(y) \right\} ds.$$

PROOF. We first note that for all $a > 0$,

$$\nu^s(B(a)^c) \leqslant \mu^s(B(a/2)^c) + M_X(B(a/2)^c) < \infty$$

and

$$\int_{B(a)} (\|y\|_2)^2 \, d\nu^s(y) \leqslant \int_{B(a)} (\|y\|_2)^2 \, d\mu^s(y) + \int_{B(a)} (\|y\|_2)^2 \, dM_X(y) < \infty.$$

Now for any semigroup of operators $(T_t | t > 0)$ with infinitesimal generator A, the formula

$$(4.8) \qquad\qquad (T_t - I) = \int_0^t A T_s \, ds$$

is well known (see [6, p. 308]). When we interpret (4.8) in our context we conclude that for any $f \in C_\infty(R^N)$ we have

$$(4.9) \quad \begin{aligned} &\int_{R^N} (f(z) - f(z - x)) \, d\mu^t(x) \\ &\qquad = \int_0^t \left\{ \int_{R^N} (f(z) - f(z - y)) - (f'(z) \cdot \tau(y)) \, d\nu^s(y) \right\} ds. \end{aligned}$$

Setting $z = 0$ in (4.9) gives us (4.7) for $f \in C_\infty(R^N)$, after we note that ν^s is symmetric so $\int f'(z) \cdot \tau(y) \, d\nu^s = 0$.

Suppose now that f is bounded, measurable and constant on $B(a)$ for some $a > 0$. Let ν be the measure $\int_0^t \nu^s \, ds$. We note that

$$\nu(B(a)^c) \leqslant \int_0^t \left(\mu^s(B(a/2)^c) + M_X(B(a/2)^c) \right) ds \leqslant t \left(1 + M_X(B(a/2)^c) \right).$$

Let f_n be a sequence of bounded infinitely differentiable functions which are constant on $B(a)$ and such that $f_n(x) - f_n(0) \to f(x) - f(0)$ in $L^1(\nu + \mu^t)$. Since (4.7) is valid for f_n we conclude it is valid for f.   Q.E.D.

COROLLARY 4.1. *Let f be a bounded measurable real valued function which is constant in a neighbourhood of* 0. *Let* $F(s) = \int (f(0) - f(-x)) \, d\mu^s(x)$. *Then* F *is a continuous and differentiable function for* $s \geqslant 0$ *with*

$$F'(s) = \int (f(0) - f(-x)) \, d\nu^s(x).$$

PROOF. This follows immediately from (4.7).

THEOREM 4.1. *Let X be a symmetric infinitely divisible random vector in* R^N. *Suppose the Levy-Khinchine measure* M_X *is in* U_N; *then X is a symmetric unimodal random vector in* R^N.

PROOF. Suppose first the special case that $M_X(R^N)$ is a finite constant and let $\|M_X\|$ denote this constant. Let μ_X stand for the distribution of X. It is well known that μ_X can be written as the mixture

$$(4.10) \qquad \mu_X = \exp^{(-\|M_X\|)} \sum_{k=0}^{\infty} \frac{1}{k!} \, (M_X)^k$$

where $(M_X)^k$ stands for the k-fold convolution of M_X with itself. (See [9, p. 79].) For convenience rewrite (4.10) in symbolical form as

$$(4.11) \qquad \mu_X = \exp^{(-\|M_X\|)} \exp^{M_X}.$$

Now each measure $(M_X)^k$ is in U_N by Theorem 3.2 (which clearly extends to finite measures in U_N). Since U_N is closed under mixtures we conclude μ_X is in U_N in the special case $\|M_X\| < \infty$.

For the case with $\|M_X\| = \infty$ we consider the representation

$$(4.12) \qquad M_X = \int_{W_N} \lambda_K \, d\nu_X(\lambda_K)$$

which follows from (2.1), where ν_X is a measure on W_N. For any $r > 0$ we know that $F(r) \equiv M_X(B(r)^c) < \infty$. We conclude from (4.12) that for any $a > 0$ we have $\nu_X\{\lambda_K | \lambda_K(B(r)^c) > 1/a\} \leqslant aF(r)$. Let us now define a sequence of finite measures ν_n on W_N by

$$(4.13) \qquad \nu_n(A) = \nu_X\Big(A \cap \big\{\lambda_K | \lambda_K\big(B(1/n)^c\big) \geqslant 1/2\big\}\Big).$$

Let M_n be the measures on R^N defined by

$$M_n = \int_{W_N} \lambda_K \, d\nu_n(\lambda_K).$$

Clearly M_n are finite measures in U_N with $M_n(R^N) \leqslant 2F(1/n)$.

It is trivial to see that for all $a > 0$ we have $M_n \to^w M$ on $B(a)^c$. Using [9, p. 80] we then see that $X_n \to^D X$ where X_n is the symmetric infinitely divisible random vector with Levy-Khinchine measure M_n. Now X_n is symmetric unimodal by the special case first considered in the proof of this theorem. We conclude X is symmetric unimodal by applying Theorem 3.1.   Q.E.D.

COROLLARY 4.2. *Let X be a symmetric infinitely divisible random vector in R^N whose Levy-Khinchine measure M_X factors as $\theta \times \Psi$, where θ is a measure on $(0, \infty)$ while Ψ is a finite measure on S_{N-1}. Suppose further that θ has finite density $q(t)$ which is a decreasing function of t. Then X is symmetric unimodal.*

PROOF. We claim that M_X is in U_N. For any $s \in S_{N-1}$ let $\hat{s}$ stand for the set $\{s, -s\}$ and let $\hat{S}_{n-1}$ stand for the image of S_N under the map $s \to \hat{s}$. Let $\hat{\Psi}$ stand for the image of the measure Ψ under this map. Finally for any $\hat{s} \in \hat{S}_{N-1}$, let $p_{\hat{s}}$ stand for the measure on R^N defined by

(4.14) $p_{\hat{s}}(B) = \theta\{t|ts \in B\} + \theta\{t| - ts \in B\}$

where B is any Borel subset of R^N and $\hat{s} = \{-s, s\}$. Clearly we can write M_X as a mixture

(4.15) $M_X = \int_{\hat{S}_{N-1}} p_{\hat{s}} \, d\hat{\Psi}(\hat{s}).$

Now each measure $p_{\hat{s}}$ is symmetric and clearly has support on the line $L = \{ts|t \in R\}$. In other words, since U_N is closed under mixing, we have reduced our claim to the one-dimensional case, where it is obvious. We conclude M_X is indeed in U_N, and hence it follows from Theorem 4.1 that X itself is symmetric unimodal.   Q.E.D.

We are now going to apply our results to the study of multivariate stable densities.

DEFINITION 4.2. A random vector X in R^N is said to be stable of index α if whenever $X_1, \ldots, X_n$ are independent random vectors distributed like X, then for some nonrandom vector V_n we have that $n^{-1/\alpha}(X_1 + \cdots + X_n + V_n)$ is distributed like X. It can be shown that α must be in $(0, 2]$ and that X is multivariate normal if $\alpha = 2$. (See Feller [4] for some basic facts about stable random variables.)

It can be shown that every stable random vector is infinitely divisible. If X is a symmetric stable random vector in R^N of index α then its characteristic function $\phi_X(v)$ has the representation

(4.16) $- \log(\phi_X(v)) = \int_{S_{N-1}} |(v \cdot s)|^\alpha \, d\Psi(s)$

where Ψ is a finite measure on S_{N-1} (see [12]). We can rewrite (4.16) as

(4.17) $\log(\phi_X(v)) = k_\alpha \int_{S_{N-1}} \int_0^\infty (\cos(v \cdot ts) - 1) \frac{1}{t^{1+\alpha}} \, dt \, d\Psi(s)$

for some positive constant k_α, so we conclude that the Levy-Khinchine measure for X factors as in Corollary 4.2, whence it follows that X is symmetric and unimodal.

We now explore the notion of stochastic dominance in the context of symmetric infinitely divisible random vectors and derive results which are new even in one dimension (where they are easier to prove).

THEOREM 4.2. *Let $(X_i(t)|t > 0)$ be a time homogeneous stochastic process with symmetrically distributed independent increments with values in R^N for $i = 1, 2.$ (Assume also that $X_i(0) = 0$ a.s.) Let M_i be the Levy-Khinchine measure for $X_1(1)$ and suppose $M_i \in U_N$ for $i = 1, 2.$ Then $M_1 \succ M_2$ if and only if $X_1(t) \succ X_2(t)$ for all $t > 0$.*

PROOF. If $X_1(t) \succ X_2(t)$ for all $t > 0$, then applying Corollary 4.1 to functions $f \in \mathcal{F}_N$ which are constant in a neighbourhood of 0, we conclude that

$$(4.18) \qquad \int_{R^N}(f(0) - f(x))\, dM_1(x) > \int_{R^N}(f(0) - f(x))\, dM_2(x)$$

for such f. To show that (4.18) holds for any $f \in \mathcal{F}_N$ choose $0 < \theta < f(0)$ and define

$$(4.19) \qquad f_\theta(x) = \begin{cases} f(0) - \theta & \text{if } f(0) > f(x) > f(0) - \theta, \\ f(x) & \text{for all other } x. \end{cases}$$

It is clear that f_θ is constant in a neighbourhood of 0, so (4.18) is valid for f_θ. Noting that $f_\theta(0) - f_\theta(x) \uparrow f(0) - f(x)$ as $\theta \to 0$, we conclude that (4.18) is valid for all $f \in \mathcal{F}_N$, so $M_1 \succ M_2$.

We now argue the reverse implication. Suppose $M_1 \succ M_2$ and let $\theta > 0$. We shall show that $X_1((1 + \theta)t) \succ X_2(t)$ for all $t > 0$. (Note that $X_1(1 + \theta)$ has Levy-Khinchine measure $(1 + \theta)M_1$.) We denote by μ_i^t the distribution of $X_i(t)$ and by ν_i^t the convolution of μ_i^t with M_i for $i = 1, 2$.

Suppose now that $f \in \mathcal{F}_{R^N}$. By [3] we have that

$$(4.20) \qquad f(x) \leqslant \exp(-c\|x\|_2) \quad \text{on } \{x|\ \|x\|_2 > d\}$$

for some $c, d > 0$. From (4.20) we conclude

$$(4.21) \qquad \int_L f(x)\, d\operatorname{Vol}_L(x) < \infty$$

for any linear subspace L of R^N.

Consider now the inequality

$$(4.22) \qquad \int_{R^N}(f(0) - f(x))\, d\mu_1^{t(1+\theta)}(x) > \int_{R^N}(f(0) - f(x))\, d\mu_2^t(x)$$

which we wish to establish for all $f \in \mathcal{F}_N$ and all $t \geqslant 0$. Our procedure will be to establish (4.22) for successively larger classes of functions by a step-wise procedure. Let us make the following definitions: For $\delta > 0$, $\mathcal{F}_\delta$ stands for $\{f|f \in \mathcal{F}_N, \int_{R^N}(f(0) - f(x))\, dM_1(x) > \delta\}$. For $\delta, a > 0$, $\mathcal{F}_{\delta,a}$ stands for the set $\{f|f \in \mathcal{F}_\delta, f \equiv 1 \text{ on } B(a)\}$. For $\delta, a, k > 0$, $\mathcal{F}_{\delta,a,k}$ stands for the set $\{f|f \in \mathcal{F}_{\delta,a}, |f(x) - f(y)| \leqslant k\|x - y\|_2 \text{ for } x, y \in R^N\}$. First we claim that if (4.22) holds for all elements of $\mathcal{F}_\delta$ for any $\delta > 0$, then (4.22) holds for all elements of $\mathcal{F}_N$. This is obvious since for any nontrivial M_1, it is certainly true that $\int_{R^N}(f(0) - f(x))\, dM_1(x) > \delta > 0$ for any $f \in \mathcal{F}_N$ and δ sufficiently small. Next we claim that if (4.22) holds for all elements of $\mathcal{F}_{\delta,a}$ for any $\delta, a > 0$ then (4.22) holds for all elements of $\mathcal{F}_{\delta'}$, for any $\delta' > 0$. This follows since for $f \in \mathcal{F}_{\delta'}$, $0 < \theta < 1$, and a sufficiently small, we have $B(a) \subset \{x|f(x) > 1 - \theta\}$. Defining f_θ as in (4.19), we have $f_\theta \in \mathcal{F}_{\delta'/2,a}$ for θ

sufficiently small (a depending on θ). Now assuming (4.22) valid for $\mathcal{F}_{\delta'/2,a}$ we conclude (4.22) is valid for f by letting $\theta \to 0$.

Our last claim is that if (4.22) holds for $\mathcal{F}_{\delta,a,k}$ for any δ, a, $k > 0$ then (4.22) holds for $\mathcal{F}_{\delta',a'}$ for any δ', $a' > 0$. We show this by first letting $h(x)$ be a fixed element of $\mathcal{F}_N$ such that $h(x) \equiv 1$ on $B(1/2)$ and $h(x) \equiv 0$ on $B(1)^c$. We may also assume that for some $k_0 > 0$ we have $|h(x) - h(y)| \leqslant k_0 \|x - y\|_2$ for $x, y \in R^N$. We define $h_n(x) = (n)^N h(nx)$ and note that $|h_n(x) - h_n(y)| \leqslant (n)^{N+1} k_0 \|x - y\|$ for $x, y \in R^N$. Suppose now that $f \in \mathcal{F}_{\delta',a'}$ for δ', $a' > 0$. We define $f_n = f * h_n$ (which exists by 4.21). It is clear that $f_n(x) \to f(x)$ for all $x \in R^N$, and that $f_n \in \mathcal{F}_N$. (We know that f is a logarithmically concave function by [3] or [10].) It is also clear that f_n is identically 1 on $B(a'/2)$ for n sufficiently large. Noting that $\sup_n |f_n(0) - f_n(x)| \leqslant f(0)$ for all x we conclude that

$$\int_{R^N} (f_n(0) - f_n(x)) \, dM_1(x) \to \int_{R^N} (f(0) - f(x)) \, dM_1(x)$$

by the Lebesgue dominated convergence theorem. It follows that

$$\int_{R^N} (f_n(0) - f_n(x)) \, dM_1(x) > \delta'/2$$

for n sufficiently large. Since $|f_n(x) - f_n(y)| \leqslant (n)^{N+1} k_0 \|x - y\|_2$ for $x, y \in R^N$, we have that f_n is in $\mathcal{F}_{\delta'/2,a'/2,(n)^{N+1}k_0}$. So if we assume (4.22) valid for all $\mathcal{F}_{a,\delta,k}$ we conclude (4.22) is valid for f by letting $n \to \infty$.

At this point we have convinced ourselves that it is sufficient to demonstrate (4.22) for $f \in \mathcal{F}_{\delta,a,k}$ for δ, a, $k > 0$. Suppose by way of contradiction that (4.22) is false for some $t > 0$ and some $f \in \mathcal{F}_{\delta,a,k}$. Then the set $V = \{t | t > 0 \text{ and } (4.22) \text{ is false for some } f \text{ in } \mathcal{F}_{\delta,a,k}\}$ is nonempty. Let $T = \inf(V)$. If $T = 0$ then there exists a sequence $t_N \downarrow 0$ and $f_n \in \mathcal{F}_{\delta,a,k}$ such that (4.22) fails for f_n at t_n.

Now by [9, p. 51] we know that for $i = 1, 2$,

(4.23)
$$\lim_{t \to 0} \sup_{f \in \mathcal{F}_{\delta,a,k}} \left| \frac{1}{t} \int_{R^N} (f(0) - f(x)) \, d\mu_i^t(x) \right.$$
$$\left. - \int_{R^N} (f(0) - f(x)) \, dM_i(x) \right| = 0.$$

We conclude from (4.23) and the fact that (4.22) is false for t_n and f_n that

(4.24)
$$0 < \delta < \limsup_{n \to \infty} \int_{R^N} (f_n(0) - f_n(x))(1 + \theta) \, dM_1(x)$$
$$\leqslant \liminf_{n \to \infty} \int_{R^N} (f_n(0) - f_n(x)) \, dM_2(x).$$

Clearly (4.24) contradicts the assumption that $M_1 \succ M_2$; we conclude that $T > 0$.

It is easy to show that $\mu_1^{s(1+\theta)} \succ \mu_2^s$ for all $s \in (0, T)$ if T is finite by arguing as above.

We will show that $T = \infty$. Suppose not, then for some $f \in \mathcal{F}_{\delta,a,k}$ we have (4.22) valid for $t \in [0, T)$ but fails when $t = T$. Let

$$F_1(s) = \int_{R^N} (f(0) - f(-x)) \, d\mu_1^{s(1+\theta)}(x)$$

and let

$$F_2(s) = \int_{R^N} (f(0) - f(-x)) \, d\mu_2^s(x).$$

We know that $F_i(s)$ are continuous and differentiable functions by Corollary 4.1. We have already shown that $F_1'(0) > F_2'(0)$. It follows that $F_1(s) > F_2(s)$ for $s \in [0, t_1)$ for some $t_1 > 0$. Using Corollary 4.1 again, and noting that $\mu_1^{s(1+\theta)} \succ \mu_2^s$ for $s \in (0, T)$ we see that $F_1(T) > F_2(T)$ which is a contradiction. Q.E.D.

5. Other notions of unimodality and some counterexamples. In this section we consider other possible definitions of unimodality for symmetric random vectors in R^N. We end with a counterexample relating to symmetric stable random vectors.

The earliest relevant definition is due to Anderson [1] who defined a random variable X in R^N to be symmetric unimodal if it had a density function f with respect to Vol_N such that f was symmetric and the sets $\{x \mid f(x) > \mu\}$ were convex for all $\mu \in [0, \infty)$. This notion of unimodality is not closed under convolution, as he himself was able to show.

In [13] Sherman introduced a more general definition. According to him a random variable X in R^N was symmetric unimodal if X had a density function f with respect to Vol_N such that $\lim_{n\to\infty}\|f - f_n\|_1 = 0$, where f_n was some finite mixture $\sum_1^{k_N} \alpha_i \lambda_{K_i}$ with K_i compact, symmetric, convex sets in R^N containing 0 in their interior, while $\|h\| = \max\{\|h\|_\infty, \|h\|_1\}$. ($\| \ \|_\infty$ is the supremum norm and $\| \ \|_1$ is the $L^1(\text{Vol}_N)$ norm.) This notion of unimodality is closed under convolution, but it is easy to see that it is not closed under weak convergence.

A more recent definition is due to Olshen and Savage [8]. They define a random vector X in R^N to be α-unimodal about 0 if for all real valued bounded, nonnegative, Borel functions g on R^N the function $t^\alpha E(g(tx))$ is decreasing as t increases, $t \in [0, \infty)$. If X has a density function with respect to Vol_L for some linear subspace L of R^N of dimension k, then according to [6] X is α-unimodal about 0 if and only if there exists some version f of the density such that the function $(sx)^{k-\alpha}f(sx)$ is decreasing for any $x \in L$ as s increases, $s \in [0, \infty)$. The notion of α-unimodality is closed under weak convergence, but not under convolution (the convolution of α-unimodal

probability measures with β-unimodal probability measures is only $\alpha + \beta$-unimodal).

Comparing the above notions of unimodality to our own we see clearly that any symmetric random vector X which is unimodal according to Anderson or Sherman is unimodal in our sense as well. On the other hand if a symmetric random vector X in R^N has distribution μ which is a mixture of measures all in U_L where L is a (variable) linear subspace of R^N with $\dim(L) \leqslant k$, then we easily conclude that X is k-unimodal.

For symmetric random vectors X in R^N with density with respect to Vol_N the condition that X be N-unimodal (i.e. that some version of the density of X is decreasing along rays from the origin) seems to be the most general notion of unimodality possible. It is clear that any symmetric random vector in R^N which is unimodal in our sense is N-unimodal. We now give an example showing the converse is not true.

EXAMPLE 5.1. Let $K = \text{conv}(0, 0), (1, 1), (1, -1)$, let $A = K \cup - K$, and let $f = I_A / \text{Vol}_2(A)$. It is clear that f is 2-unimodal. To show that $f \, d\, \text{Vol}_2$ is not in U_2 we consider the function

$$F(s) = \int_{-1/2}^{+1/2} \int_{-1/2}^{+1/2} f(x + s, y) \, dx \, dy.$$

We evaluate $F(0) = \frac{1}{4}$ and $F(\frac{1}{2}) = \frac{1}{2}$. However, if f is in U_2 then it is clear from Corollary 2.1 that $F(s)$ is decreasing as s increases, $s \in [0, \infty)$.

In [5] Ghosh introduces another definition of multivariate unimodality (called type 1) by defining a random vector X in R^N to be unimodal with vertex at 0 if for every $a \in R^N$ we have $a \cdot X$ has a one-dimensional unimodal distribution with vertex at 0 (i.e. concave to the left of 0 and convex to the right of 0). He claimed his definition was motivated by geometrical considerations, however, as the following example shows, densities which are unimodal in the Ghosh sense need not even be N-unimodal.

EXAMPLE 5.2. For any $\varepsilon \in (0, 1)$ consider the density $f(x, y)$ in R^2 defined by $f(x, y) = K_\varepsilon (I_{B(1)} + \varepsilon(x^2 + y^2))$, where K_ε is a normalizing constant. A trivial calculation shows that all one-dimensional marginals of f are symmetric unimodal, hence f is unimodal in the Ghosh sense.

We conclude with an instructive example of a stable density.

EXAMPLE 5.3. Consider the function

$$f(x, y) = \frac{1}{\pi^2} \left(\frac{1}{1 + x^2} \right) \left(\frac{1}{1 + y^2} \right).$$

It is the density of a bivariate Cauchy random vector, both coordinates independent and identically distributed, and gives an example of a random vector which is symmetric unimodal in our sense, but is not unimodal according to Anderson's definition. (Note that the curves $(1 + x^2)(1 + y^2) =$

C are starlike in shape, but not convex, if C is large.)

ADDED IN PROOF. R. Ahmad and A. M. Abonammoh have announced the solution of a conjecture of Sherman which provides another characterization of the class of symmetric unimodal random vectors. (See I. M. S. Bulletin **6** (1977), 38.) Dharmadhikari and Jogdeo [Ann. Math. Statist. **4** (1976), 607–613] have done related work.

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